Improved Benders decomposition approach to complete robust optimization in box-interval

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ABSTRACT

Robust optimization is based on the assumption that uncertain data has a convex set as well as a finite set termed uncertainty. The discussion starts with determining the robust counterpart, which is accomplished by assuming the indeterminate data set is in the form of boxes, intervals, box-intervals, ellipses, or polyhedra. In this study, the robust counterpart is characterized by a box-interval uncertainty set. Robust counterpart formulation is also associated with master and subproblems. Robust Benders decomposition is applied to address problems with convex goals and quasiconvex constraints in robust optimization. For all data parameters, this method is used to determine the best resilient solution in the feasible region. A manual example of this problem's calculation is provided, and the process is continued using production and operations management–quantitative methods (POM-QM) software.

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1. INTRODUCTION

Many optimization problems in practice face obstacles such as the fact that the available data is not known exactly when the data problem should be solved. This can be due to measurement errors, modeling errors, or a lack of information available when making decisions [1]. Stochastic optimization and robust optimization are two methodologies that have been proposed to cope with the issue of data uncertainty in the optimization problem. In stochastic optimization, the optimization problem contains expectations and probability constraints where the assumptions are a combined probability distribution of known uncertain parameters. However, in practice, optimization stochastic does not have sufficient data to estimate its distribution, especially in large-scale problems [2].

Initially developed in the early 1970 s, robust optimization was intended to give decision makers with a framework where probability models could not be worked out [3]. To discover a solution that is viable for all potential data parameters up to the worst case, robust optimization assumes that the uncertain data have a convex set and a boundary set called the uncertainty set [4], [5]. Specifically, robust optimization plays an important role in things that are easy to do and provides an overview of the solution to the worst case that efficiently meets the requirements i) high resilience and protection against uncertainty and ii) achieving objective values that are close to the problem objectives [6]. One of the well-known methods of dealing with data uncertainty in optimization problems is the robust counterpart method proposed by [7]. A semi-infinite optimization problem—one with a limited number of variables and an infinite constraint function—is the formulation that is addressed when uncertain data are included in the model under consideration [6].

Ben-Tal *et al*. [4], robust formulation can be achieved by assuming the uncertain data set is in three sets which are expressed as an uncertain set in the form of boxes, intervals, ellipses, or polyhedral [7]–[9]. Ben-Tal and Nemirovski [10], to analyze the computationally robust counterpart analysis, the robust counterpart must be converted into linear programming (LP), conic quadratic optimization (CQO), or semidefinite optimization (SDO). With this research, the Benders decomposition approach is used to solve robust optimization problems for linear programming in a box-interval state in the presence of quasiconvex constraints. Where the Benders decomposition method is expected to be able to solve a complex set of variables that must be solved in the master problem [11]–[13].

- Problem description

Integer programming problems, such as those posed by Benders decomposition, are well-documented in the literature, and it was used to tackle a variety of robust optimization problems, including Beale integer programming problems [14]. The Benders decomposition is used by Saito and Murota [15] to tackle issues involving robust mix integer programming with ellipsoidal uncertainty. Syed *et al*. [16] considered Benders decomposition to define which of the accessible power plants to reduce price of power. Poojari [17], using Benders decomposition to get a robust solution in supply chain planning problems. Karamyar [18], using Benders decomposition for location problems and machine allocation in health care schedules. Emami [19], developed a robust counterpart model in the problem of receiving orders and scheduling in machines. Order issues are determined by different date, revenue, late penalty, processing time and setup time on the machine. From the proposed model, several valid pieces are introduced to accelerate the convergence of the Benders algorithm and heuristic methods to get a feasible solution.

From the various literatures described above, this study discusses solving optimization problems for quasiconvex constraints in a box-interval uncertainty. In this problem, robust counterpart is expressed as a semi-infinite linear programming problem which is an optimization problem with many variables and infinitely many constraints [20]. This method will create a strong companion model of the proposed model by introducing several valid parts to accelerate the convergence of the classical Benders algorithm and heuristic methods to get a feasible solution.

2. THE PROPOSED METHOD

2.1. Robust optimization

The formula for the indefinite linear optimization is as follows:

$$\min_x \{c^T x : Ax \leq d\}_{(c,A,d)\in U} \tag{1}$$

where $c \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}$ and $d \in \mathbb{R}^m$ is an uncertain coefficient, and U shows the user the uncertainty set. The basic paradigm of robust optimization is based on the following three assumptions [3], [4], [21]: i) all decision variables $x \in \mathbb{R}^n$ represents the here and now decision, where the decision variable must get a certain numerical value as a result of solving the problem before the actual data, ii) it is the responsibility of the decision maker, if the real data falls within a predetermined uncertainty range U, to make a decision. Iii) the constraint of the uncertainty problem is hard, i.e., the decision maker cannot abide the violation of the constraint when the data is in a predetermined uncertainty set U.

2.2. Robust counterpart

Assume $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ is certified, then the robust reformulation of (1) is called the robust counterpart problem which as shown in [4], [22]:

$$\min_x \{c^T x : C(\zeta)x \leq q, \forall \zeta \in \mathbb{Z}\} \tag{2}$$

where $\mathbb{Z} \subset \mathbb{R}^L$ shows the primitive uncertainty set. A solution $x \in \mathbb{R}^n$ is called robust feasible if it satisfies the uncertain constraint $C(\zeta)x \leq q$ for all realization $\zeta \in \mathbb{Z}$.

2.3. Box-interval uncertainty

In box-interval uncertainty U_I, an optimization problem can be formulated as a semi-infinite program with robust counterparts [23]:

$$\min z \tag{3}$$

$s.t. f(x,\zeta) \leq z, \forall\zeta \in U_I$

$g(x,\zeta) \leq 0, \forall\zeta \in U_I$

$x \in X$

In (3), there is $U_\infty = \{\zeta \| \zeta \|_\infty \leq \psi\}$ [6] where ψ is a control parameter in the uncertainty set [24] as illustrated in Figure 1. Based on [25] the value of $\psi \leq 1$.

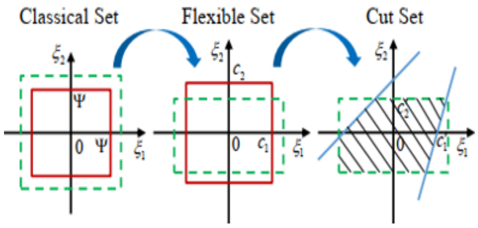


Figure 1. Box-interval illustration

2.4. Convergence in robust optimization

Definition 1: convex function: a function $g(p,\hat{p})$ is called quasiconvex on $\hat{p} \in [-\Delta p, +\Delta p]$ if and only if $\hat{p} \in [-\Delta p, +\Delta p], g(p,\hat{p}) \leq \max\{g(p,\Delta p), g(p,-\Delta p)\}$ for all p [20], [26].

Definition 2: objective robustness: n the case of a candidate points (p^c, q^c) objective robustness predominates if the inequality is lowered:

$$\frac{f(p^c,q^c,\hat{p},\hat{q}) - f(p^c,q^c,0,0)}{\varepsilon} \leq 1 \quad (4)$$

fulfill for all $\hat{p} \in [-\Delta p, \Delta p]$ and $\hat{q} \in [-\Delta q, \Delta q]$.

Definition 3: Robustness of feasibility: In the case of a candidate solution (p^c, q^c) if:

$$g_j(p,q,\hat{p},\hat{q}) \leq 0, \forall j = 1,\dots,J \quad (5)$$

fulfill for all $\hat{p} \in [-\Delta p, \Delta p]$ and $\hat{q} \in [-\Delta q, \Delta q]$, then the robust feasibility is prevailed.

Assume that the objective function is convex with the constraint convex so that the solution persists when nominal $\hat{p} = \hat{q} = 0$.

$$\min_{p,q} f(p,q) \quad (6)$$

$s.t. g(p,q,\hat{p},\hat{q}) \leq 0, j = 1,\dots,J$

$p \in R^n, q \in Z^m, \hat{p} \in R^{n_u}, \hat{q} \in R^{m_u}$

$\forall\hat{p} \in [-\Delta p, \Delta q], \forall\hat{q} \in [-\Delta q, \Delta q]$

Definition 4: robust points: points that meet robust eligibility for (6) are considered robust points. S_R represents the set of robust points for (6).

Definition 5: locally optimal robust: for the robust optimization problem, the locally optimal robust $p*$ solution is a robust point such that there is a set of neighbors U of the optimal $p*$ robust solution $f(p*) \leq f(p), \forall p \in U$.

Definition 6: globally optimal robust: for formulation (6), globally optimally robust (p^*, q^*) is an optimal robust point on $f(p^*, q^*) \leq f(p,q), \forall(p,q) \in S_R$.

3. METHOD

3.1. Robust optimization re-formulation

By adding an auxiliary function α_b (6) will produce two master problems for the defined Benders decomposition:

$$\alpha_b(\hat{p},\hat{q}) = \min_{p,q} f(p,q) \tag{7}$$

$$\text{s.}t.\ g_j(p,q,\hat{p},\hat{q}) \leq 0, j = 1, \dots, J$$

$$p \in R^n, q \in Z^m$$

$$\alpha_q(q) = \min_{p} f(p,q) \tag{8}$$

$$\text{s.}t.\ g_j(p,q,\hat{p},\hat{q}) \leq 0, j = 1, \dots, J$$

$$\hat{p} \in \hat{p}_{fixed}, \hat{q} = \hat{q}_{fixed}$$

$$p \in R^n$$

3.2. Algorithm improvement

By following the basic algorithm of [27]–[29], the following Benders decomposition improvement algorithm is used to solve problems (7) and (8):

Step 0. Set the I iteration to 0 with the smallest constant as the tolerance value.

Step 1. Set iteration for $I = ite + 1$ where variables $(\hat{p},\hat{q})$ are a complex variable that is fixed in the subproblem. So, the master problem becomes:

$$\min_{\alpha_q,q} \alpha_q + f_q(q)$$

$$s.t.\ q_{down} \leq q \leq q_{up}$$

$$\alpha_q \leq \alpha_q^{\min} \tag{9}$$

$$s.t. - \Delta p \leq \hat{p} \leq \Delta p$$

$$-\Delta p \leq \hat{q} \leq \Delta q$$

$$\alpha_q \leq \alpha_q^{\max} \tag{10}$$

the limits of α_p, α_q determined and given $\alpha_p = \alpha_p^{ite}, \alpha_q = \alpha_q^{ite}$ and $q = q_{fixed}^{ite}, \hat{p} = \hat{p}_{fixed}^{ite}, \hat{q} = \hat{q}_{fixed}^{ite}$.

Step 2. Fix complicated variable values $\hat{p}$, then solve the subproblem with Bender's standard decomposition method.

$$w = \min_{p} f_p(p) \tag{11}$$

$$\text{s.}t.\ g_j(p,q,\hat{p},\hat{q}) \leq 0, j = 1, \dots, J$$

$$q = q_{fixed}^{ite}, (dual = \lambda^{ite})$$

$$\hat{p} = \hat{p}_{fixed}^{ite}$$

$$\hat{q} = \hat{q}_{fixed}^{ite}$$

Step 3. Check for convergence. Set $z_{sub} = w$, $z_{mast1} = \alpha_p^{ite}, z_{mast2} = \alpha_q^{ite}$. If the difference $z_{up} - z_{down1} \leq \varepsilon$ and $z_{up} - z_{down2} \leq \varepsilon$ then the iteration stops.

Step 4. Add standard Benders snippet for master problem (9).

$$\alpha_q \geq f\left(p_{ite}^{sol}, q_{ite}^{sol}, \hat{p}_{ite}^{sol}, \hat{q}_{ite}^{sol}\right) + \lambda^{ite}\left(q - q_{ite}^{sol}\right) \tag{12}$$

Step 1 (return). N represents the number of iterations that have been completed. Once Benders pieces have been added, solve the following master problem,

$$\min_{\alpha_q,q} \alpha_q + f_q(q).t\ q_{down} \leq q \leq q_{up} \tag{13}$$

$$\alpha_p \geq f\left(p_{ite}^{sol}, q_{ite}^{sol}, \hat{p}_{ite}^{sol}, \hat{q}_{ite}^{sol}\right) + \lambda^{ite}\left(q - q_{ite}^{sol}\right)$$

for $ite = 1, \ldots, N$

$$\alpha_q \geq \alpha_q^{min}$$

return to step 2 and continue with iterations until convergence is met.

4. RESULTS AND DISCUSSION

Numerical calculations are show the improvement algorithm for the Benders decomposition method to obtain a robust solution to problems (7) and (8). In this case, the robust optimization problem lies in the box-interval uncertain constraint.

$$\min_{p,q} z = -\frac{1}{4}p - q \tag{14}$$

$$s.t.\ (-1+\zeta_1)p + (-1+\zeta_2)q \leq 5$$

$$\left(-\frac{1}{2}+\zeta_3\right)p + (1+\zeta_4)q \leq \frac{15}{2}$$

$$\left(\frac{1}{2}+\zeta_4\right)p + (1+\zeta_5)q \leq \frac{35}{2}$$

$$(1+\zeta_6)p + (-1+\zeta_7)q \leq 10$$

$$(1+\zeta_8)p \leq 16$$

$$(1+\zeta_9)q \geq 0$$

$$\forall \zeta_i \in box - interval, i = 1, \ldots, I$$

4.1. Solution

By taking $\zeta \in [-0.1, +0.1]$ and $\zeta_i = 0.1$ which is substituted into the problem constraint (14), problem (14) will become a deterministic problem (15) which is robust by forming a robust counterpart into the set of box-interval uncertainties:

$$\min_{p,q} z = -\frac{1}{4}p - q \tag{15}$$

$$s.t.\ -0.9p - 0.9q \leq 5$$

$$-0.4p + 1.1q \leq 7.5$$

$$0.6p + 1.1q \leq 17.5$$

$$1.1p - 0.9q \leq 10$$

$$1.1p \leq 16$$

$$1.1q \geq 0$$

the next stage, the deterministic optimization problem (15) is solved by the improvement algorithm Benders Decomposition starting with:

Step 0: iteration $v = 1$. The master problem,

$$\underset{p,\alpha}{\text{minimize}}\ -\frac{1}{4}p + \alpha$$

$$s.t.\, 1.1p \leq 16$$

$$-25 \leq \alpha$$

Step 1: the subproblem solution:

$$\min_{q} z = -q$$

$$s.t. -0.9p - 0.9q \leq 5$$

$$-0.4p + 1.1q \leq 7.5$$

$$0.6p + 1.1q \leq 17.5$$

$$1.1p - 0.9q \leq 10$$

$$1.1p \leq 16\ (dual = \lambda)$$

$$1.1q \geq 0$$

The solution is $S^{(1)} = \left(p^{(1)}, q^{(1)}\right) = (14.5,8)$ by minimizing the objective function z. And the optimal value of the dual variable with the constraint $p^{(1)} = 14$ is $\lambda^{(1)} = 0.5$.
Step 2: check for convergence. The upper and lower limits of the optimal objective function,

$$z_{up}^{(1)} = -\frac{1}{4}p^{(1)} - q^{(1)} = -\frac{1}{4}(14.5) - 8 = -11.625$$

$$z_{down}^{(1)} = -\frac{1}{4}p^{(1)} + \alpha^{(1)} = -\frac{1}{4}(14.5) - 25 = -28.625$$

because of the difference $z_{up}^{(1)} - z_{down}^{(1)} = 17 > \varepsilon$, then the process continues to the next step.
Step 3: master problem solution. Update the iteration $v = 1 + 1 = 2$. Master problem:

$$\underset{p,\alpha}{\text{minimize}}\ -\frac{1}{4}p + \alpha$$

$$s.t.\, 8.7 + 1.1(p - 14) \leq 17.5$$

$$1.1p_1 \leq 16$$

$$\alpha \leq 17.5$$

where the first constraint is done by Benders cut 1 which is related to the previous iteration.

Because too many iterations have been carried out and the convergence has not been achieved, the problem solving (15) is continued with the help of the production and operations management–quantitative methods (POM-QM) software. By using the POM-QM software, the convergence results are obtained with the upper and lower limits of the optimal objective function being:

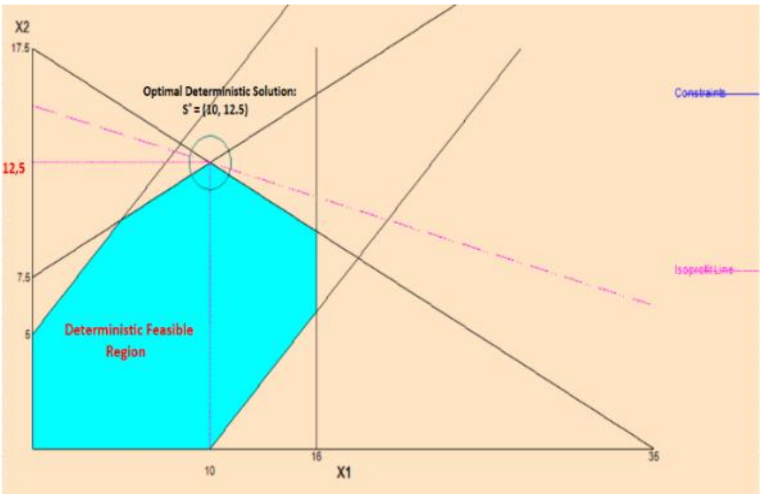
$$z_{up}^{(5)} = -\frac{1}{4}p^{(5)} - q^{(5)} = -12.95$$

$$z_{down}^{(5)} = -\frac{1}{4}p^{(5)} + \alpha^{(5)} = -12.95$$

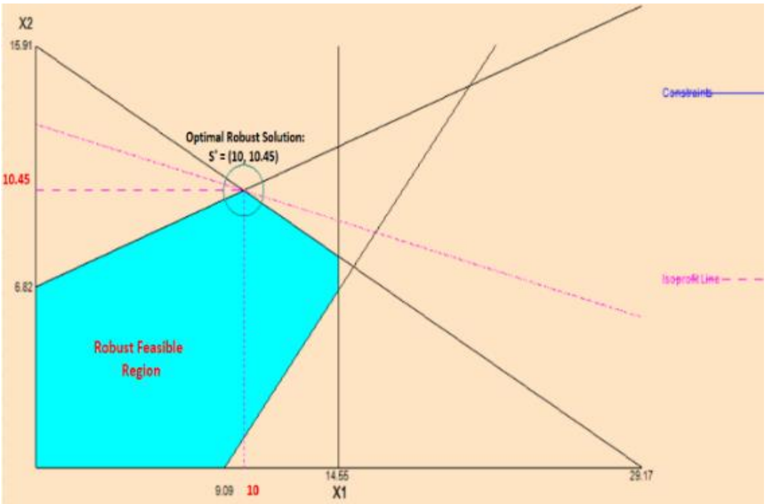
because of the difference $z_{up}^{(5)} - z_{down}^{(5)} = -12.95 + 12.95 = 0 < \varepsilon$ then the optimal solution is obtained by $p* = 10, q* = 10.45, z* = -12.95$. The results of the iteration summary with the improvement of the Benders decomposition method are shown in Table 1. Figures 2(a) and 2(b) provide examples of the problem's feasible and robust feasible regions problem (14).

Table 1. Optimal solution and iteration improvement of Benders decomposition

Iteration	$\boldsymbol{p}^{(v)}$	$\boldsymbol{q}^{(v)}$	$\boldsymbol{\alpha}^{(v)}$	$\boldsymbol{\lambda}^{(v)}$	$\boldsymbol{z}_{up}^{(v)}$	$\boldsymbol{z}_{down}^{(v)}$
1	14.5	8	-25	0.5	-11.625	-28.625
2	0	4.5	-17.5	-1	-4.5	-17.5
3	4.5	8.5	-13.33	-0.5	-9.625	-14.455
4	-6	0	-5.25	-0.5	-1.5	-6.75
5	10	10.45	-10.45	-0.5	-12.95	-12.95



(a)



(b)

Figure 2. Provide examples of the (a) illustration of feasible region and (b) illustration of robust feasible region

The deterministic problem (14) is a linear programming problem by giving the value of $p* = 10, q* = 12.5$ and $z* = -15$. It is possible to establish the robust optimization (14) solution by examining the corner point of the robust workable area, which provides the value of $p* = 10, q* = 10.45$ and $z* = -12.95$. Thus, as illustrated in Table 2, the globally optimal interval is met.

Table 2. Comparison of feasible regions with robust feasible regions

Description	Deterministic solution	Robust solution
$\boldsymbol{p}*$	10	10
$\boldsymbol{q}*$	12.5	10.45
$\boldsymbol{z}*$	-15	-12.95

5. CONCLUSION

To create a stochastic optimization model, one assumes that the input data is known precisely and is equivalent to a predetermined nominal value. Data uncertainty has a significant influence on the model's quality and feasibility, which this technique does not consider. When data are obtained from various nominal values, certain constraints may be broken and the best solution established using nominal data may no longer be optimal or feasible. Problems that include parameters or decision variables with box-interval uncertainty may be solved using this method. It is suggested that Bender's decomposition improvement approach be used to produce a robust optimal linear programming solution.

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